

## Computing the Primordial Power Spectra Directly

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### ABSTRACT

The tree order power spectra of primordial inflation depend upon the norm-squared of mode functions which oscillate for early times and then freeze in to constant values. We derive simple differential equations for the power spectra, that avoid the need to numerically simulate the physically irrelevant phases of the mode functions. We also derive asymptotic expansions which should be valid until a few e-foldings before first horizon crossing, thereby avoiding the need to evolve mode functions from the ultraviolet over long periods of inflation.

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# 1 Introduction

Cosmological perturbations from primordial inflation [1, 2, 3] have a crucial importance for fundamental theory because they are first quantum gravitational effect ever detected [4, 5], because they provide strong evidence for primordial inflation [6, 7, 8], and because they set the initial conditions for structure formation in cosmology [9, 10, 11]. Two sorts of perturbations are produced during single-scalar inflation: a scalar perturbation characterized by the field  $\zeta(t, \vec{x})$ , and a tensor perturbation characterized by the transverse-traceless field  $h_{ij}(t, \vec{x})$ . The scalar signal has been imaged in the anisotropies of the cosmic microwave background [12, 13], and by measuring the matter power spectrum with large scale structure surveys [14]. The tensor signal has not been imaged so far [12, 13] but strenuous efforts are underway to detect it through the polarization of the cosmic microwave background [15, 16, 17, 18].

The two perturbation fields are defined on the homogeneous, isotropic and spatially flat geometry characterized by scale factor  $a(t)$ , with Hubble parameter  $H(t)$  and slow roll parameters  $\epsilon(t)$  and  $\eta(t)$ ,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x}, \quad H(t) \equiv \frac{\dot{a}}{a}, \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2}, \quad \eta(t) \equiv \epsilon - \frac{\dot{\epsilon}}{2H\epsilon}. \quad (1)$$

Only the quadratic parts of their Lagrangians are relevant to current measurements (see [19] for an explanation of the full formalism),

$$\mathcal{L}_\zeta^{(2)} = \frac{\epsilon a^3}{8\pi G} \left\{ \dot{\zeta}^2 - \frac{\partial_k \zeta \partial_k \zeta}{a^2} \right\}, \quad (2)$$

$$\mathcal{L}_h^{(2)} = \frac{a^3}{64\pi G} \left\{ \dot{h}_{ij} \dot{h}_{ij} - \frac{\partial_k h_{ij} \partial_k h_{ij}}{a^2} \right\}. \quad (3)$$

The reported results [12, 13, 14] for the scalar and tensor power spectra are consistent with evaluating the following 2-point correlators long after the first horizon crossing time  $t_k$  such that  $k = H(t_k) a(t_k)$ ,

$$\Delta_{\mathcal{R}}^2(t, k) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left\langle \Omega \left| \zeta(t, \vec{x}) \zeta(t, \vec{0}) \right| \Omega \right\rangle, \quad (4)$$

$$\Delta_h^2(t, k) \equiv \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left\langle \Omega \left| h_{ij}(t, \vec{x}) h_{ij}(t, \vec{0}) \right| \Omega \right\rangle. \quad (5)$$

At tree order the correlators (4-5) can be expressed in terms of the scalar mode function  $v(t, k)$  and its tensor cousin  $u(t, k)$ ,

$$\Delta_{\mathcal{R}}^2(t, k) = \frac{k^3}{2\pi^2} \times 4\pi G \times |v(t, k)|^2 + O(G^2) , \quad (6)$$

$$\Delta_h^2(t, k) = \frac{k^3}{2\pi^2} \times 32\pi G \times 2 \times |u(t, k)|^2 + O(G^2) . \quad (7)$$

The relevant mode equations and normalization conditions are,

$$\ddot{v} + \left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{v} + \frac{k^2}{a^2}v = 0 \quad , \quad W_v \equiv v\dot{v}^* - \dot{v}v^* = \frac{i}{\epsilon a^3} , \quad (8)$$

$$\ddot{u} + 3H\dot{u} + \frac{k^2}{a^2}u = 0 \quad , \quad W_u \equiv u\dot{u}^* - \dot{u}u^* = \frac{i}{a^3} . \quad (9)$$

The initial conditions for the mode functions (which correspond to the choice of Bunch-Davies vacuum) derive from the WKB solutions in the far ultraviolet (that is, for  $k \gg H(t)a(t)$ ),

$$v(t, k) \longrightarrow \frac{\exp[-ik \int^t dt' / a(t')]}{\sqrt{2ka^2(t)\epsilon(t)}} \left\{ 1 + O\left(\frac{Ha}{k}\right) \right\} , \quad (10)$$

$$u(t, k) \longrightarrow \frac{\exp[-ik \int^t dt' / a(t')]}{\sqrt{2ka^2(t)}} \left\{ 1 + O\left(\frac{Ha}{k}\right) \right\} . \quad (11)$$

As is evident from these asymptotic forms, both mode functions oscillate and fall off in the far ultraviolet. They become approximately constant near the time of first horizon crossing  $t_k$ . (One can infer the existence of constant solutions quite generally when the  $\frac{k^2}{a^2(t)}$  terms in equations (8-9) become irrelevant.) It is these constant amplitudes which determine the crucial theoretical predictions for the power spectra through relations (6-7).

One of the frustrating things about this formalism is the need to employ approximation techniques, even for evaluating tree order formulae such as (6-7), because equations (8-9) for the mode functions cannot be solved analytically for general scale factor  $a(t)$  [20, 21]. Examples of such approximation techniques include [22, 23, 24]:

- Assuming the slow roll parameter  $\epsilon(t)$  is constant;
- Matching the leading ultraviolet and infrared forms at  $t = t_k$ ; and

- Employing the full WKB solutions.

When greater accuracy is needed, one must resort to numerical evolution of (8-9) from the known initial conditions (10-11) until well past the time of first horizon crossing [23]. Excellent numerical solution techniques exist [25, 26, 27], but a large fraction of their power is wasted on reproducing the oscillations of the mode functions, which contribute nothing to the power spectra. This can be especially time-consuming for models in which there is an extended phase of inflation, or when scanning properties of classes of models.

A more effective technique was developed recently for the tensor power spectrum [28] in order to work out the gravitational wave signal from a novel model of inflation in which the Hubble parameter oscillates for a brief time [29, 30]. The key to the new technique is to convert relations (9) into an equation for the norm-squared of the mode function,  $M(t, k) \equiv |u(t, k)|^2$ . This is the quantity that actually enters the tensor power spectrum (7), and it has a much more sedate evolution than the mode function. Evolving  $M(t, k)$  avoids the wasted effort of numerically simulating the oscillations of  $u(t, k)$ . Further, an excellent asymptotic expansion can be derived for  $M(t, k)$  which must be valid until only a few e-foldings before first horizon crossing, no matter how long inflation persists [28]. It is therefore only necessary to numerically evolve  $M(t, k)$  for a few e-foldings, from just before  $t_k$  until it becomes constant.

The purpose of this paper is to extend this more effective technique to the scalar mode functions using a trick [21] for converting solutions to  $u(t, k)$  into solutions for  $v(t, k)$ . To further simplify the formalism, we express our results in terms of the tree order power spectra (6-7) directly, without ever mentioning the mode functions. Section 2 derives the key equations for  $\Delta_{\mathcal{R}}^2(t, k)$  and  $\Delta_h^2(t, k)$ , and section 3 gives their asymptotic expansions. Our discussion comprises section 4.

## 2 Differential Equations for $\Delta_{\mathcal{R}}^2$ and $\Delta_h^2$

We begin by reviewing how one derives an equation for  $M(t, k) \equiv |u(t, k)|^2$ . The first step is to write out the first and second time derivatives,

$$\dot{M}(t, k) = i\dot{u}^* + u\dot{u}^* , \quad (12)$$

$$\ddot{M}(t, k) = \ddot{u}u^* + 2i\dot{u}\dot{u}^* + u\ddot{u}^* . \quad (13)$$

One next eliminates  $\ddot{u}$  and  $\ddot{u}^*$  in (13) by using (9), and then recognizing factors of  $M$  and  $\dot{M}$ ,

$$\ddot{M}(t, k) = -3H\dot{M} - \frac{2k^2}{a^2}M + 2\dot{u}\dot{u}^* . \quad (14)$$

The final factor involving  $\dot{u}\dot{u}^*$  can be expressed in terms of  $M$  by subtracting the square of the Wronskian  $W_u$  in expression (9) from the square of  $\dot{M}$  in expression (12),

$$\dot{M}^2 - W_u^2 = 2M \times 2\dot{u}\dot{u}^* = \dot{M}^2 + \frac{1}{a^6} \implies 2\dot{u}\dot{u}^* = \frac{\dot{M}^2}{2M} + \frac{1}{2Ma^6} . \quad (15)$$

Substituting (15) in relation (14) gives the desired equation for  $M(t, k)$ ,

$$\ddot{M} + 3H\dot{M} + \frac{2k^2}{a^2}M = \frac{\dot{M}^2}{2M} + \frac{1}{2Ma^6} . \quad (16)$$

Deriving an equation for  $N(t, k) \equiv |v(t, k)|^2$  entails the same first step,

$$\dot{N}(t, k) = \dot{v}v^* + v\dot{v}^* , \quad (17)$$

$$\ddot{N}(t, k) = \ddot{v}v^* + 2\dot{v}\dot{v}^* + v\ddot{v}^* . \quad (18)$$

We next employ expression (8) to eliminate  $\ddot{v}$  and  $\ddot{v}^*$  in favor of  $N$  and  $\dot{N}$ ,

$$\ddot{N} = -\left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{N} - \frac{2k^2}{a^2}N + 2\dot{v}\dot{v}^* . \quad (19)$$

Subtracting the squares of the  $\dot{N}$  from (17) and the Wronskian  $W_v$  in equation (8) gives  $\dot{v}\dot{v}^*$ ,

$$\dot{N}^2 - W_v^2 = 2N \times 2\dot{v}\dot{v}^* = \dot{N}^2 + \frac{1}{\epsilon^2 a^6} \implies 2\dot{v}\dot{v}^* = \frac{\dot{N}^2}{2N} + \frac{1}{2N\epsilon^2 a^6} . \quad (20)$$

And the final equation for  $N(t, k)$  comes from substituting (20) in (19),

$$\ddot{N} + \left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{N} + \frac{2k^2}{a^2}N = \frac{\dot{N}^2}{2N} + \frac{1}{2N\epsilon^2 a^6} . \quad (21)$$

Because equations (16) and (21) are nonlinear, they incorporate the Wronskian normalization as well as the evolution equations for the mode

functions. Converting them to equations for  $\Delta_{\mathcal{R}}^2(t, k)$  and  $\Delta_h^2(t, k)$  requires only changing the coefficients of the final terms,

$$\ddot{\Delta}_{\mathcal{R}}^2 + \left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{\Delta}_{\mathcal{R}}^2 + \frac{2k^2}{a^2}\Delta_{\mathcal{R}}^2 = \frac{(\dot{\Delta}_{\mathcal{R}}^2)^2}{\Delta_{\mathcal{R}}^2} + \frac{2G^2k^6}{\pi^2\epsilon^2a^6}\frac{1}{\Delta_{\mathcal{R}}^2}, \quad (22)$$

$$\ddot{\Delta}_h^2 + 3H\dot{\Delta}_h^2 + \frac{2k^2}{a^2}\Delta_h^2 = \frac{(\dot{\Delta}_h^2)^2}{\Delta_h^2} + \frac{2^9G^2k^6}{\pi^2a^6}\frac{1}{\Delta_h^2}. \quad (23)$$

### 3 Asymptotic Expansions for $\Delta_{\mathcal{R}}^2$ and $\Delta_h^2$

Equations (22)-(23) require initial values for the power spectra and their first time derivatives. Of course these derive from the ultraviolet limits (10-11), which imply the zeroth order terms for an expansion in powers of  $(Ha/k)^2$ ,

$$\Delta_{\mathcal{R}}^2(t, k) \longrightarrow \frac{Gk^2}{\pi\epsilon a^2} \left\{ 1 + O\left(\frac{H^2a^2}{k^2}\right) \right\}, \quad (24)$$

$$\Delta_h^2(t, k) \longrightarrow \frac{16Gk^2}{\pi a^2} \left\{ 1 + O\left(\frac{H^2a^2}{k^2}\right) \right\}. \quad (25)$$

We shall develop the expansion for  $\Delta_h^2(t, k)$  to second order, and then employ a trick for converting tensor mode functions to scalar ones [21] to obtain the corresponding expansion of  $\Delta_{\mathcal{R}}^2(t, k)$ . These expansions are so accurate that there seems little point to numerically evolving for anything except the last few e-foldings before first horizon crossing.

From the asymptotic limit (25) one can see that the “big” terms in the evolution equation (23) are those with no derivatives. It is best to segregate these terms,

$$\Delta_h^2 - \left(\frac{16\pi Gk^2}{\pi a^2}\right)^2 \frac{1}{\Delta_h^2} = \frac{a^2}{2k^2} \left[ -\ddot{\Delta}_h^2 - 3H\dot{\Delta}_h^2 + \frac{(\dot{\Delta}_h^2)^2}{2\Delta_h^2} \right]. \quad (26)$$

The desired expansion is now easy to read off [28],

$$\begin{aligned} \Delta_h^2(t, k) = & \frac{16Gk^2}{\pi a^2} \left\{ 1 + \left[1 - \frac{1}{2}\epsilon\right] \left(\frac{Ha}{k}\right)^2 \right. \\ & \left. + \left[ \frac{9}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{3}{4}\epsilon^3 + \left(\frac{7}{4} - \frac{3}{4}\epsilon\right) \frac{\dot{\epsilon}}{H} + \frac{1}{8} \frac{\ddot{\epsilon}}{H^2} \right] \left(\frac{Ha}{k}\right)^4 + O\left(\frac{H^6a^6}{k^6}\right) \right\}. \end{aligned} \quad (27)$$

Note that this expansion is in powers of  $(\frac{Ha}{k})^2$ , rather than the expansion in powers of  $\frac{Ha}{k}$  one gets for the mode function. The better convergence for small  $\frac{Ha}{k}$  is one more indication of the superiority of our method.

The trick for converting  $\Delta_h^2(t, k)$  into  $\Delta_{\mathcal{R}}^2(t, k)$  involves simultaneously changing the scale factor and the meaning of time [21],

$$a(t) \longrightarrow \sqrt{\epsilon(t)} \times a(t) , \quad (28)$$

$$\frac{\partial}{\partial t} \longrightarrow \frac{1}{\sqrt{\epsilon(t)}} \times \frac{\partial}{\partial t} . \quad (29)$$

One can see that these two replacements convert the  $u(t, k)$  mode relations (9) into the relations (8) for  $v(t, k)$ . Replacements (28-29) also convert the  $\Delta_h^2(t, k)$  equation (23) into expression (22) for  $\Delta_{\mathcal{R}}^2(t, k)$ . We could have used them to derive that equation had the straightforward derivation not been so trivial.

To convert the asymptotic expansion (27) into an expansion for  $\Delta_{\mathcal{R}}^2(t, k)$  we first note that the replacements (28-29) imply the following geometrical replacements,

$$H \longrightarrow \frac{H}{\sqrt{\epsilon}} \times [1 + \epsilon - \eta] \equiv \frac{H}{\sqrt{\epsilon}} \times D , \quad (30)$$

$$\epsilon \longrightarrow \frac{(1 + \epsilon - \eta)(2\epsilon - \eta) - (\frac{\dot{\epsilon} - \dot{\eta}}{H})}{(1 + \epsilon - \eta)^2} \equiv \frac{N}{D^2} , \quad (31)$$

$$\frac{\dot{\epsilon}}{H} \longrightarrow \frac{D\dot{N} - 2N\dot{D}}{HD^4} , \quad (32)$$

$$\frac{\ddot{\epsilon}}{H^2} \longrightarrow \frac{D^2\ddot{N} - 4D\dot{N}\dot{D} - 2DN\ddot{D} + 6N\dot{D}^2}{H^2D^6} - \frac{(\epsilon - \eta)(D\dot{N} - 2N\dot{D})}{HD^5} . \quad (33)$$

After many tedious manipulations the result is,

$$\begin{aligned} \Delta_{\mathcal{R}}^2(t, k) = & \frac{Gk^2}{\pi\epsilon a^2} \left\{ 1 + \left[ 1 - \frac{1}{2}(1 + \epsilon - \eta)(2\epsilon - \eta) - \left( \frac{\dot{\epsilon} - \dot{\eta}}{2H} \right) \right] \left( \frac{Ha}{k} \right)^2 \right. \\ & + \left[ \frac{3}{8}(1 + \epsilon - \eta)(3 - \epsilon - \eta)(2 - \eta)(2\epsilon - \eta) + \left[ \frac{5}{4} + \frac{9}{2}\epsilon - \frac{17}{4}\eta - \frac{7}{8}\epsilon^2 - \frac{3}{2}\epsilon\eta + \frac{13}{8}\eta^2 \right] \frac{\dot{\epsilon}}{H} \right. \\ & + \left[ \frac{1}{2} - \frac{21}{8}\epsilon + \frac{13}{8}\eta + \epsilon^2 + \frac{1}{2}\epsilon\eta - \frac{3}{4}\eta^2 \right] \frac{\dot{\eta}}{H} + \frac{7}{4} \left( \frac{\dot{\epsilon}}{H} \right)^2 - \frac{25}{8} \left( \frac{\dot{\epsilon}}{H} \right) \left( \frac{\dot{\eta}}{H} \right) + \frac{11}{8} \left( \frac{\dot{\eta}}{H} \right)^2 \\ & \left. \left. - \left[ \frac{3}{2} + \frac{3}{8}\epsilon - \frac{3}{4}\eta \right] \frac{\ddot{\epsilon}}{H^2} + \left[ \frac{13}{8} + \frac{\epsilon}{2} - \frac{7}{8}\eta \right] \frac{\ddot{\eta}}{H^2} - \left( \frac{\ddot{\epsilon} - \ddot{\eta}}{8H^3} \right) \right] \left( \frac{Ha}{k} \right)^4 + O\left( \frac{H^6 a^6}{k^6} \right) \right\} . \quad (34) \end{aligned}$$

Unless  $\epsilon(t)$  or  $\eta(t)$  is changing rapidly on the Hubble scale, the fractional error in (34) should be of order  $(\frac{Ha}{k})^6$ . Had we used this formula to estimate  $\Delta_{\mathcal{R}}^2(t, k)$  for  $t$  as close as one e-folding before first horizon crossing, the fractional error would only be about  $2.5 \times 10^{-3}$ ; for two e-foldings that would fall to about  $6.1 \times 10^{-6}$ ; and the fractional error would only be about  $1.5 \times 10^{-8}$  at three e-foldings before horizon crossing. This is comparable to what loop corrections might give [31].

## 4 Discussion

We have derived nonlinear, second order differential equations (22) and (23) for the correlators  $\Delta_{\mathcal{R}}^2(t, k)$  and  $\Delta_h^2(t, k)$  whose late time limits give the scalar and tensor power spectra at tree order. Numerically evolving these correlators is more economical than evolving the mode functions (8-9) because one no longer has to keep track of the phases which drop out of the power spectra. This economy shows up in the asymptotic expansions (34) and (27) pertinent to the ultraviolet regime of  $k \gg H(t) a(t)$ . Our expansions for the correlators are in powers of the small parameter  $(\frac{Ha}{k})^2$ , whereas the analogous expansions for the mode functions are only in powers of  $\frac{Ha}{k}$ . In fact our asymptotic expansions (34-27) should be so accurate that they could be used to provide the initial value data for  $\Delta_{\mathcal{R}}^2(t, k)$  and  $\Delta_h^2(t, k)$  only a few e-foldings before the time  $t_k$  of first horizon crossing.

We have employed “Hubble parametrization”, in which the scale factor  $a(t)$  is assumed known and results are expressed in terms of background geometrical quantities  $H(t)$ ,  $\epsilon(t)$  and  $\eta(t)$  defined in expression (1). This is especially useful for the tensor power spectrum because gravitational waves depend only on the background geometry, no matter what caused it. Many people prefer “potential parametrization”, in which the scalar potential  $V(\phi)$  is assumed known (with the field starting at rest from some fiducial value) and results are expressed as derivatives of the potential. Approximate conversion formulae are,

$$H^2 \simeq \frac{8}{3}\pi G V \quad , \quad \epsilon \simeq \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2 \quad , \quad \eta \simeq \frac{1}{16\pi G} \left[ 2 \frac{V''}{V} - \left( \frac{V'}{V} \right)^2 \right] . \quad (35)$$

Finally, we note that the same technology can be applied to derive an equation for the power spectrum of any field whose plane wave mode functions obey the Mukhanov equation [21]. Of course the associated asymptotic



expansion is also easy to develop. Both the equation and its solution in the regime of  $k \gg H(t) a(t)$  can be obtained from the analogous tensor relations, (23) and (27), by changing the scale factor and the time as we did in equations (28-29).

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